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About O Level

ADDITIONAL MATHEMATICS

(TOPICAL)

About Thinking Process

In solving mathematical problems, we always work backward. After indentifying our main target, we go 'backward' to look for the 'easier' targets until we are able to solve the problems.

Thinking process reveals how the teacher actually goes about solving a sum in the above-said manner.

About Teacher's Comments

It reveals the extra but relevant information which is not required as part of the solutions but are extremely useful in knowing how the solutions are arrived.

period	**2009 to 2024**
contents	**June & November, Paper 1 & 2, Worked Solutions**
form	**Topic By Topic**
compiled for	**O Levels**
special features	**Thinking Process, Teacher's Comments**

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CONTENTS

Revised Syllabus

Topic 1

Functions

1 *(N09/P2/Q6)*

A function f is defined by $f : x \mapsto e^{x-1}$, where $x > 0$.

(i) State the range of f. [1]

(ii) Find an expression for f^{-1}. [2]

(iii) State the domain of f^{-1}. [1]

Thinking Process

(i) To find range ✎ find f(x) for $x > 0$.

(ii) To find f^{-1} ✎ let $y = f(x)$ and express x in terms of y.

(iii) ✎ Domain of $f^{-1}(x)$ = Range of f(x).

Solution

(i) Range of f: $f(x) > \frac{1}{e}$ **Ans.**

(ii) $f(x) = e^{x-1}$

Let $f(x) = y$

$\Rightarrow \quad y = e^{x-1}$

$\ln y = \ln e^{x-1}$

$\ln y = x - 1$

$x = 1 + \ln|y| \qquad \boxed{\because x = f^{-1}(y)}$

$\therefore \quad f^{-1}(y) = 1 + \ln|y|$

or $\quad f^{-1}(x) = 1 + \ln|x|$ **Ans.**

(iii) Domain of $f^{-1}(x)$ = range of $f(x)$

$\therefore \quad$ domain of f^{-1}: $x > \frac{1}{e}$ **Ans.**

2 *(J10/P2/Q12or)*

OR

The functions f, g and h are defined, for $x \in \mathbb{R}$, by

$$f(x) = x^2 + 1,$$
$$g(x) = 2x - 5,$$
$$h(x) = 2^x.$$

(i) Write down the range of f. [1]

(ii) Find the value of gf(3). [2]

(iii) Solve the equation $fg(x) = g^{-1}(15)$. [5]

(iv) On the same axes, sketch the graph of $y = h(x)$ and the graph of the inverse function $y = h^{-1}(x)$, indicating clearly which graph represents h and which graph represents h^{-1}. [2]

Thinking Process

(iii) To solve $fg(x) = g^{-1}(15)$ ✎ find $g^{-1}(15)$.

(iv) Note that $h^{-1}(x)$ is the reflection of $h(x)$ in the line $y = x$.

Solution

(i) $f(x) = x^2 + 1$

$\therefore$ range of $f(x)$ is: $f(x) \geq 1$ **(Ans).**

(ii) $gf(3) = g(3^2 + 1)$

$= g(10)$

$= 2(10) - 5 = 15$ **(Ans).**

(iii) Let $g(x) = y \quad \Rightarrow \quad x = g^{-1}(y)$

$\therefore \quad y = 2x - 5 \quad \Rightarrow \quad 2x = y + 5 \quad \Rightarrow \quad x = \frac{y+5}{2}$

$\Rightarrow \quad g^{-1}(y) = \frac{y+5}{2}$

$\therefore \quad g^{-1}(x) = \frac{x+5}{2}$

also, $\quad fg(x) = f(2x - 5)$

$= (2x-5)^2 + 1$

given that, $\quad fg(x) = g^{-1}(15)$

$\Rightarrow \quad (2x-5)^2 + 1 = \frac{(15)+5}{2}$

$\Rightarrow \quad 4x^2 - 20x + 25 + 1 = 10$

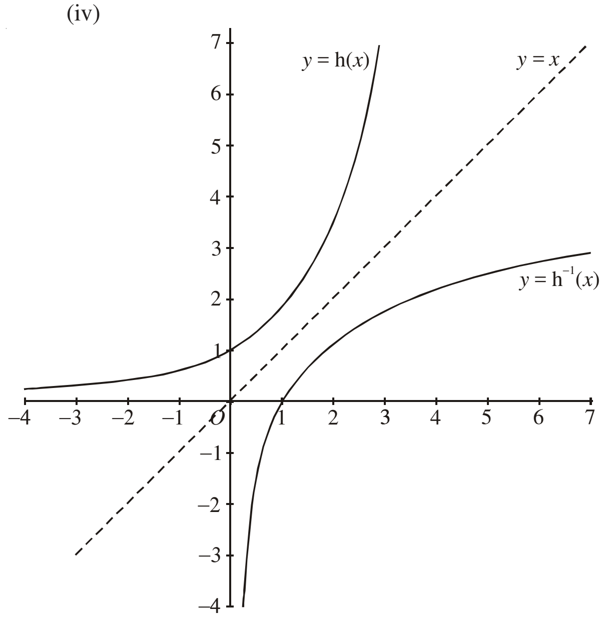
$\Rightarrow \quad 4x^2 - 20x + 16 = 0$

$\Rightarrow \quad x^2 - 5x + 4 = 0$

$\Rightarrow \quad (x-1)(x-4) = 0$

$\Rightarrow \quad x = 1 \quad$ or $\quad x = 4$ **(Ans).**

(iv)



3

(N10/P2/Q12 Either)

The functions f and g are defined, for $x > 1$, by

$$f(x) = (x+1)^2 - 4,$$

$$g(x) = \frac{3x+5}{x-1}.$$

Find

(i) fg(9), [2]

(ii) expressions for $f^{-1}(x)$ and $g^{-1}(x)$, [4]

(iii) the value of x for which $g(x) = g^{-1}(x)$. [4]

Thinking Process

(i) Find g(9). Substitute the value of g(9) into f.

(ii) Let f(x) = y and make x the subject of formula.

(iii) Find $g^{-1}(x)$ and solve.

Solution

(i) $f(x) = (x+1)^2 - 4,$

$$g(9) = \frac{3(9)+5}{9-1} = \frac{32}{8} = 4$$

$$fg(9) = f(4)$$
$$= (4+1)^2 - 4$$
$$= 25 - 4 = 21 \quad \textbf{Ans.}$$

(ii) $f(x) = (x+1)^2 - 4$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\therefore \quad y = (x+1)^2 - 4$$
$$(x+1)^2 = y+4$$
$$x+1 = \pm\sqrt{y+4}$$
$$x = -1 \pm \sqrt{y+4}$$
$$\Rightarrow \quad f^{-1}(y) = -1 \pm \sqrt{y+4}$$
$$\therefore \quad f^{-1}(x) = -1 \pm \sqrt{x+4}$$

since $x > 1$, $\Rightarrow$ $f^{-1}(x) = -1 + \sqrt{x+4}$ **Ans.**

$$g(x) = \frac{3x+5}{x-1}$$

Let $g(x) = y \Rightarrow x = g^{-1}(y)$

$$\therefore \quad y = \frac{3x+5}{x-1}$$
$$xy - y = 3x+5$$
$$xy - 3x = 5+y$$
$$x(y-3) = 5+y$$
$$x = \frac{5+y}{y-3}$$
$$\Rightarrow \quad g^{-1}(y) = \frac{5+y}{y-3}$$
$$\therefore \quad g^{-1}(x) = \frac{5+x}{x-3} \quad \textbf{Ans.}$$

(iii) $g(x) = g^{-1}(x)$

$$\Rightarrow \quad \frac{3x+5}{x-1} = \frac{x+5}{x-3}$$
$$\Rightarrow \quad (3x+5)(x-3) = (x+5)(x-1)$$
$$\Rightarrow \quad 3x^2 - 9x + 5x - 15 = x^2 - x + 5x - 5$$
$$\Rightarrow \quad 2x^2 - 8x - 10 = 0$$
$$\Rightarrow \quad x^2 - 4x - 5 = 0$$
$$\Rightarrow \quad (x-5)(x+1) = 0$$
$$\Rightarrow \quad x = 5 \quad \text{or} \quad x = -1$$

since domain for f(x) and g(x) is $x > 1$

$\therefore$ $x = 5$ **Ans.**

4

(J11/P2/Q10)

(a) (i) Express $18 + 16x - 2x^2$ in the form $a + b(x+c)^2$, where a, b and c are integers. [3]

A function f is defined by $f : x \to 18 + 16x - 2x^2$ for $x \in \mathbb{R}$.

(ii) Write down the coordinates of the stationary point on the graph of $y = f(x)$. [1]

(iii) Sketch the graph of $y = f(x)$. [2]

(b) A function g is defined by $g : x \to (x+3)^2 - 7$ for $x > -3$.

(i) Find an expression for $g^{-1}(x)$. [2]

(ii) Solve the equation $g^{-1}(x) = g(0)$. [3]

Thinking Process

(a) (i) Apply completing the square method.

(ii) Use answer to part (i).

(iii) To sketch the graph use stationary point, y- intercepts and x- intercepts.

(b) (i) Let y = $g(x)$, express x as the subject of formula.

Solution

(a) (i)

$$18 + 16x - 2x^2$$
$$= 18 - 2x^2 + 16x$$
$$= 18 - 2(x^2 - 8x)$$
$$= 18 - 2(x^2 - 8x + 4^2 - 4^2)$$
$$= 18 - 2\left((x-4)^2 - 16\right)$$
$$= 18 - 2(x-4)^2 + 32$$
$$= 50 - 2(x-4)^2 \quad \textbf{Ans.}$$

(ii) From part (i), the coordinates of stationary point is (4, 50) **Ans.**

(iii) $y = 18 + 16x - 2x^2$

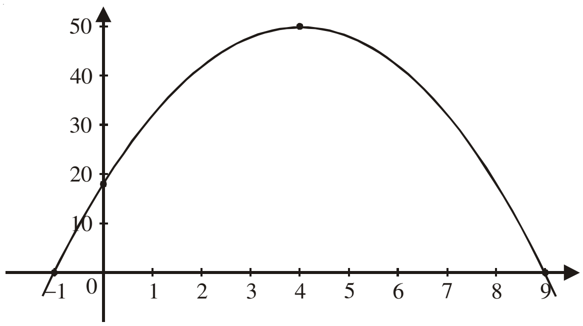
at $x = 0$, $y = 18$ $\Rightarrow$ y-intercept is (0, 18)

at $y = 0$, $18 + 16x - 2x^2 = 0$

$$x^2 - 8x - 9 = 0$$
$$(x-9)(x+1) = 0$$
$$x = 9 \quad \text{or} \quad x = -1$$

$\Rightarrow$ x-intercepts are (9, 0) & (−1, 0)

turning point is (4, 50), and it is the maximum point.



(b) (i) $g(x) = (x+3)^2 - 7$

Let $g(x) = y$ $\Rightarrow$ $x = g^{-1}(y)$

$\therefore$ $(x+3)^2 - 7 = y$

$$(x+3)^2 = y + 7$$
$$x + 3 = \pm\sqrt{y+7}$$
$$x = -3 \pm \sqrt{y+7}$$

$\Rightarrow$ $g^{-1}(y) = -3 \pm \sqrt{y+7}$

$\therefore$ for the given domain,

$g^{-1}(x) = -3 + \sqrt{x+7}$ **Ans.**

(ii) $g^{-1}(x) = g(0)$

$$-3 + \sqrt{x+7} = 2$$
$$\sqrt{x+7} = 5$$
$$x + 7 = 25$$
$$x = 18$$ **Ans.**

5 *(J12/P1/Q10)*

(a) It is given that $f(x) = \dfrac{1}{2+x}$ for $x \neq -2$, $x \in \mathbb{R}$.

(i) Find $f''(x)$. [2]

(ii) Find $f^{-1}(x)$. [2]

(iii) Solve $f^2(x) = -1$. [3]

(b) The functions g, h and k are defined, for $x \in \mathbb{R}$, by

$$g(x) = \frac{1}{x+5},\ x \neq -5,$$
$$h(x) = x^2 - 1,$$
$$k(x) = 2x + 1.$$

Express the following in terms of g, h and/ or k.

(i) $\dfrac{1}{(x^2-1)+5}$ [1]

(ii) $\dfrac{2}{x+5} + 1$ [1]

Thinking Process

(a) (i) Differentiate f(x) twice ✐ Write $\frac{1}{2+x}$ as $(2+x)^{-1}$.

(ii) Let $y = f(x)$, and express x in terms of y.

(iii) To find $f^2(x)$ ✐ find $f\,f(x)$.

(b) (i) To express $\frac{1}{(x^2-1)+5}$ ✐ substitute h(x) into g(x).

(ii) To express $\frac{2}{x+5} + 1$ ✐ substitute g(x) into k(x).

Solution

(a) (i) $f(x) = \dfrac{1}{2+x} = (2+x)^{-1}$

$$f'(x) = (-1)(2+x)^{-2} = -(2+x)^{-2}$$
$$f''(x) = 2(2+x)^{-3} = \frac{2}{(2+x)^3}$$ **Ans.**

(ii) $f(x) = \dfrac{1}{2+x}$

Let $f(x) = y$ $\Rightarrow$ $x = f^{-1}(y)$

$\therefore$ $y = \dfrac{1}{2+x}$

$$y(2+x) = 1$$
$$2y + xy = 1$$
$$xy = 1 - 2y$$
$$x = \frac{1-2y}{y}$$

$\Rightarrow$ $f^{-1}(y) = \dfrac{1-2y}{y}$

$\therefore$ $f^{-1}(x) = \dfrac{1-2x}{x}$ for $x \neq 0$ **Ans.**

(iii) $f^2(x) = -1$

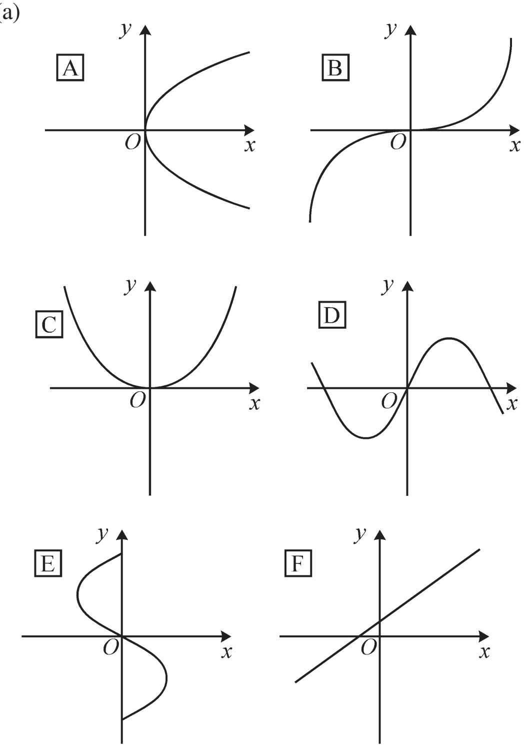
$$f\,f(x) = -1$$
$$f\left(\frac{1}{2+x}\right) = -1$$
$$\frac{1}{2+\left(\frac{1}{2+x}\right)} = -1$$
$$\frac{2+x}{4+2x+1} = -1$$
$$2 + x = -5 - 2x$$
$$3x = -7$$
$$x = -\frac{7}{3} = -2\frac{1}{3}$$ **Ans.**

(b) (i) $\dfrac{1}{(x^2-1)+5} = \text{gh}(x)$ **Ans.**

(ii) $\dfrac{2}{x+5}+1 = \text{kg}(x)$ **Ans.**

6 *(J13/P2/Q3)*

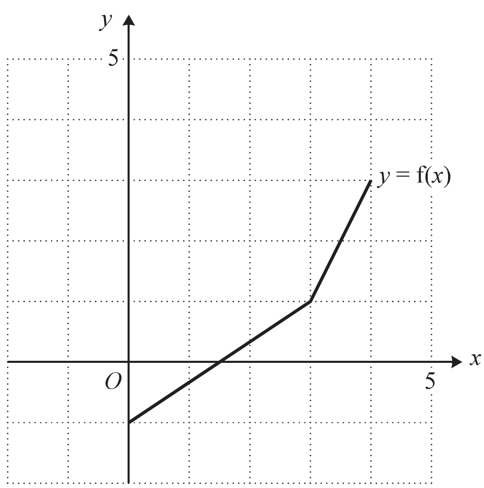
(a)



(i) Write down the letter of each graph which does **not** represent a function. [2]

(ii) Write down the letter of each graph which represents a function that does **not** have an inverse. [2]

(b)



The diagram shows the graph of a function $y = \text{f}(x)$. On the same axes sketch the graph of $y = \text{f}^{-1}(x)$. [2]

Thinking Process

(a) (i) Apply vertical line test i.e. if a vertical line intersect the curve at two or more points then it is not a function.

(ii) Apply horizontal line test i.e. if a line parallel to x-axis cuts a curve at two or more points then it is many-to-one function and has no inverse.

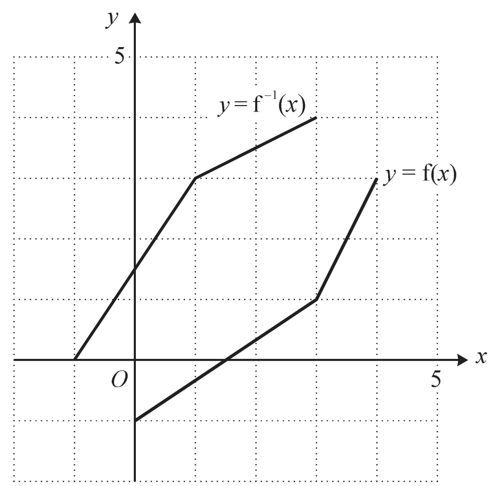
(b) $\text{f}^{-1}(x)$ is the reflection of $\text{f}(x)$ in the line $y = x$

Solution

(a) (i) A and E

(ii) C and D

(b)



7 *(N13/P1/Q12)*

(a) A function f is such that $\text{f}(x) = 3x^2 - 1$ for $-10 \le x \le 8$.

(i) Find the range of f. [3]

(ii) Write down a suitable domain for f for which f^{-1} exists. [1]

(b) Functions g and h are defined by

$$\text{g}(x) = 4\text{e}^x - 2 \quad \text{for} \quad x \in \mathbb{R},$$

$$\text{h}(x) = \ln 5x \quad \text{for} \quad x > 0.$$

(i) Find $\text{g}^{-1}(x)$. [2]

(ii) Solve $\text{gh}(x) = 18$. [3]

Thinking Process

(a) (i) To find range ⇒ equate $\text{f}'(x) = 0$ to find the y-coordinate of the turning point. Find f(–10) and f(8).

(ii) To find domain ⇒ find the set of values of x for which $\text{f}(x)$ is a 1–1 function.

(b) (i) Let $y = \text{g}(x)$. make x the subject of formula.

(ii) To solve $\text{gh}(x) = 18$ ⇒ substitute $\text{h}(x)$ into $\text{g}(x)$.

Solution

(a) (i) $f(x) = 3x^2 - 1$

$f'(x) = 6x$

for turning points, $f'(x) = 0$

$\Rightarrow \quad 6x = 0 \quad \Rightarrow \quad x = 0$

$\therefore \quad f(0) = 3(0)^2 - 1 = -1$

also, $f(-10) = 3(-10)^2 - 1 = 299$

$f(8) = 3(8)^2 - 1 = 192 - 1 = 191$

$\therefore$ range of f is: $-1 \leq f(x) \leq 299$ **Ans.**

(ii) $x \geq 0$ **Ans.**

Other suitable domain for which f^{-1} exists are:
$0 \leq x \leq 8$ or $-10 \leq x \leq 0$

(b) (i) $g(x) = 4e^x - 2$

Let $g(x) = y \quad \Rightarrow \quad x = g^{-1}(y)$

$\therefore \quad y = 4e^x - 2$

$\Rightarrow \quad e^x = \dfrac{y+2}{4}$

$\Rightarrow \quad \ln e^x = \ln\left|\dfrac{y+2}{4}\right|$

$\Rightarrow \quad x(\ln e) = \ln\left|\dfrac{y+2}{4}\right| \quad \Rightarrow \quad x = \ln\left|\dfrac{y+2}{4}\right|$

$\Rightarrow \quad g^{-1}(y) = \ln\left|\dfrac{y+2}{4}\right|$

$\therefore \quad g^{-1}(x) = \ln\left|\dfrac{x+2}{4}\right|$ **Ans.**

(ii) $gh(x) = 18$

$\Rightarrow \quad g(\ln 5x) = 18$

$4e^{\ln 5x} - 2 = 18$

$4(5x) = 20$

$20x = 20$

$x = 1$ **Ans.**

Alternative Answer:

$gh(x) = 18$

$h(x) = g^{-1}(18)$

$\ln 5x = \ln\left(\dfrac{18+2}{4}\right)$

$\ln 5x = \ln 5$

$5x = 5$

$x = 1$ **Ans.**

8 *(J14/P2/Q11)*

The functions f and g are defined, for real values of x greater than 2, by

$$f(x) = 2^x - 1,$$
$$g(x) = x(x+1).$$

(i) State the range of f. [1]

(ii) Find an expression for $f^{-1}(x)$, stating its domain and range. [4]

(iii) Find an expression for $gf(x)$ and explain why the equation $gf(x) = 0$ has no solutions. [4]

Thinking Process

(i) Substitute of $x = 2$ to find the corresponding least value of $f(x)$.

(ii) Let $y = f(x)$. make x the subject of the formula.

(iii) Substitute f(x) into g to find $gf(x)$. Equate $gf(x)$ to 0 and solve for x.

Solution

(i) $f(2) = 2^2 - 1 = 3$

$\therefore$ range of $f(x)$ is: $f(x) > 3$ **Ans.**

(ii) $f(x) = 2^x - 1$

Let $y = f(x)$

$\Rightarrow \quad y = 2^x - 1$

$2^x = y + 1$

$x = \log_2(y+1)$

$\Rightarrow \quad f^{-1}(y) = \log_2(y+1)$ $\quad \because x = f^{-1}(y)$

$\therefore \quad f^{-1}(x) = \log_2(x+1)$ **Ans.**

Domain of $f^{-1}(x)$: $x > 3$

Range of $f^{-1}(x)$: $f^{-1}(x) > 2$ **Ans.**

Domain of $f^{-1}(x)$ = Range of $f(x)$
Range of $f^{-1}(x)$ = Domain of $f(x)$

(iii) $gf(x)$

$= g(2^x - 1)$

$= (2^x - 1)(2^x - 1 + 1)$

$= 2^x(2^x - 1)$ **Ans.**

$gf(x) = 0$

$\Rightarrow \quad 2^x(2^x - 1) = 0$

$\Rightarrow \quad 2^x = 0$ (not possible) or $2^x - 1 = 0$

$2^x = 1$

$2^x = 2^0$

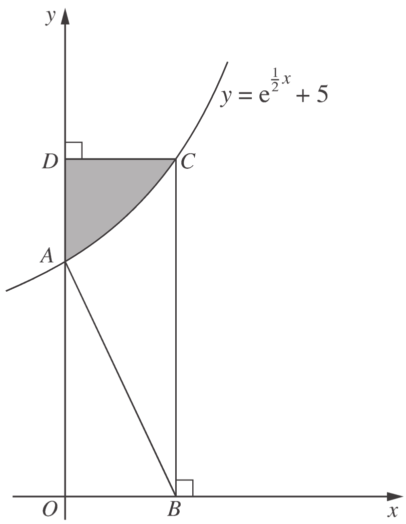
$x = 0$

Given domain for both functions is $x > 2$.
$x = 0$ is not in the given domain. Hence $gf(x) = 0$ has no solution.

Topic 18

Integration

1 *(J09/P2/Q12 Or)*



The diagram shows part of the curve $y = e^{\frac{1}{2}x} + 5$ crossing the y-axis at A. The normal to the curve at A meets the x-axis at B.

(i) Find the coordinates of B. [4]

The line through B, parallel to the y-axis, meets the curve at C. The line through C, parallel to the x-axis, meets the y-axis at D.

(ii) Find the area of the shaded region. [6]

Thinking Process

(i) Find y coordinate of A by substituting $x = 0$ in the equation of curve.
Differentiate y to find gradient of tangent at A.
Apply, $\text{grad. of normal} = \dfrac{-1}{\text{grad. of tangent}}$ and find the equation of normal AB. Note that line AB meets x-axis at B.

(ii) Area of shaded region = (area of rectangle – area under the curve).

Solution

(i) $y = e^{\frac{1}{2}x} + 5 \;\cdots\cdots\cdots(1)$

curve crosses y-axis at A

$\therefore$ at A, $x = 0$,

$\Rightarrow \quad y = e^{\frac{1}{2}(0)} + 5 = 6$

$\therefore$ coordinates of A are $(0, 6)$

differentiating eq. (1) w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}$$

gradient of tangent at A: $\dfrac{dy}{dx} = \dfrac{1}{2}e^0 = \dfrac{1}{2}$

$\Rightarrow$ gradient of normal $= -2$

equation of normal AB with gradient -2 and passing through $A(0, 6)$ is:

$y - 6 = -2(x - 0)$

$y = -2x + 6 \;\cdots\cdots\cdots(2)$

normal to the curve, cuts x-axis at B,

$\therefore$ put $y = 0$, in eq. (2)

$0 = -2x + 6 \quad \Rightarrow \quad x = 3$

$\therefore$ coordinates of B are $(3, 0)$ **Ans.**

(ii) $y = e^{\frac{1}{2}x} + 5 \;\cdots\cdots\cdots(1)$

from part (i), coordinates of $B = (3, 0)$

$\therefore$ x-coordinate of C is 3,

subst. $x = 3$, in eq. (1), $\quad y = e^{\frac{3}{2}} + 5$

$\therefore$ coordinates of C are $(3, \; e^{\frac{3}{2}} + 5)$

point D lies on y-axis and on a line parallel to x-axis which passes through C.

$\therefore$ coordinates of D are $(0, \; e^{\frac{3}{2}} + 5)$

Area of shaded region

= area of rectangle $OBCD$ – area under curve

$= (OB)(OD) - \displaystyle\int_0^3 (e^{\frac{x}{2}} + 5)\, dx$

$= 3 \times (e^{\frac{3}{2}} + 5) - \left[2e^{\frac{x}{2}} + 5x \right]_0^3$

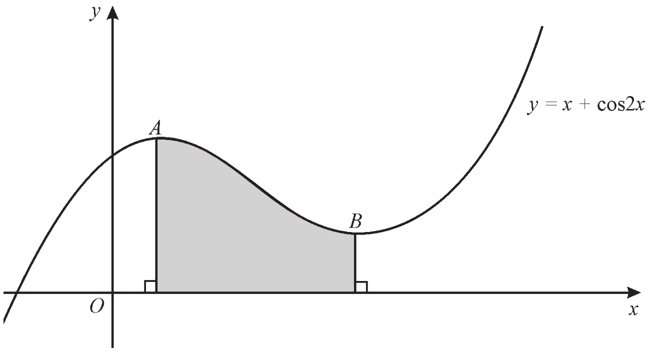
$= 3e^{\frac{3}{2}} + 15 - \left[(2e^{\frac{3}{2}} + 5(3)) - (2e^{\frac{0}{2}} + 5(0)) \right]$

$= 3e^{\frac{3}{2}} + 15 - (2e^{\frac{3}{2}} + 15 - 2)$

$= 3e^{\frac{3}{2}} + 15 - 2e^{\frac{3}{2}} - 13$

$= e^{\frac{3}{2}} + 2 = 6.48$ (3sf) **Ans.**

2 *(N09/P1/Q12 Or)*



The diagram shows part of the curve $y = x + \cos 2x$. The curve has a maximum point at A and a minimum point at B.

(i) Find the x-coordinate of the point A and of the point B. [6]

(ii) Find, in terms of π, the area of the shaded region. [5]

Thinking Process

(i) Differentiate y w.r.t. x. For stationary value, equate $\frac{dy}{dx} = 0$ and solve.

(ii) Use integration to find the area of shaded region.

Solution

(i) $y = x + \cos 2x$

$$\frac{dy}{dx} = 1 - 2\sin 2x$$

for stationary values, $\frac{dy}{dx} = 0$

$\Rightarrow \quad 1 - 2\sin 2x = 0$

$\sin 2x = \frac{1}{2}$

basic angle $\alpha = \frac{\pi}{6}$

$\therefore \quad 2x = \frac{1}{6}\pi, \ \frac{5}{6}\pi$

$\Rightarrow \quad x = \frac{1}{12}\pi, \ \frac{5}{12}\pi$

$\therefore \quad x$-coordinate of $A = \frac{1}{12}\pi$

and x-coordinate of $B = \frac{5}{12}\pi$ **Ans.**

(ii) Area of the shaded region

$$= \int_{\frac{1}{12}\pi}^{\frac{5}{12}\pi} (x + \cos 2x)\, dx$$

$$= \left[\frac{x^2}{2} + \frac{1}{2}\sin 2x \right]_{\frac{1}{12}\pi}^{\frac{5}{12}\pi}$$

$$= \left[\frac{\left(\frac{5\pi}{12}\right)^2}{2} + \frac{1}{2}\sin 2\left(\frac{5\pi}{12}\right) \right] - \left[\frac{\left(\frac{\pi}{12}\right)^2}{2} + \frac{1}{2}\sin 2\left(\frac{\pi}{12}\right) \right]$$

$$= \frac{25\pi^2}{144 \times 2} + \frac{1}{2}\sin\left(\frac{5\pi}{6}\right) - \frac{\pi^2}{144 \times 2} - \frac{1}{2}\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{25\pi^2}{288} + \frac{1}{2}\left(\frac{1}{2}\right) - \frac{\pi^2}{288} - \frac{1}{2}\left(\frac{1}{2}\right)$$

$$= \frac{25\pi^2}{288} - \frac{\pi^2}{288} + \frac{1}{4} - \frac{1}{4}$$

$$= \frac{24\pi^2}{288} = \frac{1}{12}\pi^2 \quad \textbf{Ans.}$$

3 *(N09/P2/Q6)*

(i) Given that $y = x\sqrt{4x + 12}$, show that $\frac{dy}{dx} = \frac{k(x+2)}{\sqrt{4x+12}}$, where k is a constant to be found. [4]

(ii) Hence evaluate $\int_{-2}^{6} \frac{3x+6}{\sqrt{4x+12}}\, dx$. [3]

Thinking Process

(i) Apply product rule to differentiate the given equation.

(ii) To evaluate $\int_{-2}^{6} \frac{3x+6}{\sqrt{4x+12}}\, dx$ ✐ Use the result in part (i).

Solution

(i) $y = x\sqrt{4x+12} = x(4x+12)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \sqrt{4x+12}\,\frac{d}{dx}(x) + (x)\frac{d}{dx}\sqrt{4x+12}$$

$$= \sqrt{4x+12} + x\left(\frac{1}{2}(4x+12)^{-\frac{1}{2}}\right)(4)$$

$$= \sqrt{4x+12} + \frac{2x}{\sqrt{4x+12}}$$

$$= \frac{\left(\sqrt{4x+12}\right)^2 + 2x}{\sqrt{4x+12}}$$

$$= \frac{4x+12+2x}{\sqrt{4x+12}}$$

$$= \frac{6x+12}{\sqrt{4x+12}} = \frac{6(x+2)}{\sqrt{4x+12}} \quad \textbf{Shown.}$$

$\Rightarrow \quad k = 6 \quad$ **Ans.**

(ii) $\displaystyle\int_{-2}^{6} \frac{3x+6}{\sqrt{4x+12}}\,dx = \int_{-2}^{6} \frac{3(x+2)}{\sqrt{4x+12}}\,dx$

Multiplying numerator and denomenator by 2

$$= \frac{1}{2}\int_{-2}^{6} \frac{6(x+2)}{\sqrt{4x+12}}\,dx$$

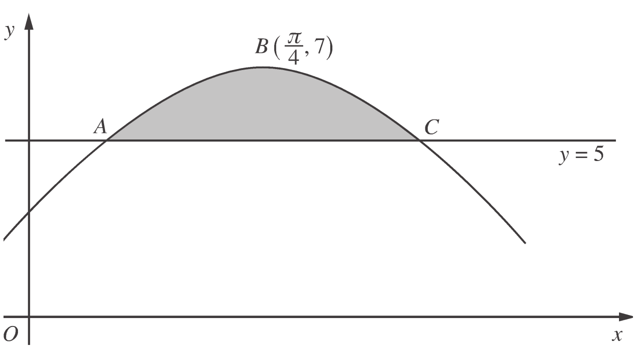
from part (i): $\dfrac{d}{dx}x\sqrt{4x+12} = \dfrac{6(x+2)}{\sqrt{4x+12}}$

$$\Rightarrow \quad \frac{1}{2}\int_{-2}^{6} \frac{d}{dx}\left(x\sqrt{4x+12}\right)dx$$

$$= \frac{1}{2}\left[x\sqrt{4x+12}\right]_{-2}^{6}$$

$$= \frac{1}{2}\left[6\sqrt{36} - (-2)\sqrt{4}\right] = \frac{1}{2}[36+4] = 20 \quad \textbf{Ans.}$$

4 *(J10/P1/Q12 Either)*



The diagram shows part of a curve for which $\dfrac{dy}{dx} = 8\cos 2x$. The curve passes through the point $B\left(\dfrac{\pi}{4}, 7\right)$.

The line $y = 5$ meets the curve at the points A and C.

(i) Show that the curve has equation $y = 3 + 4\sin 2x$. [3]

(ii) Find the x-coordinate of the point A and of the point C. [4]

(iii) Find the area of the shaded region. [5]

Thinking Process

(i) To find equation of the curve ✎ integrate the gradient.

(ii) To find the x coordinates ✎ solve the equation of line and curve simultaneously.

(iii) Area of shaded region = (area under the curve – area under the line).

Solution

(i) $\dfrac{dy}{dx} = 8\cos 2x \quad \Rightarrow \quad dy = 8\cos 2x\,dx$

integrating both sides,

$$\int dy = \int 8\cos 2x\,dx$$

$$\Rightarrow \quad y = \frac{8\sin 2x}{2} + C$$

$$\Rightarrow \quad y = 4\sin 2x + C$$

the curve passes through $B\left(\dfrac{\pi}{4}, 7\right)$

$$\Rightarrow \quad 7 = 4\sin 2\left(\frac{\pi}{4}\right) + C$$

$$\Rightarrow \quad 7 = 4\sin\frac{\pi}{2} + C \Rightarrow 7 = 4(1) + C \Rightarrow C = 3$$

$\therefore$ equation of the curve is,

$y = 4\sin 2x + 3$

or $\quad y = 3 + 4\sin 2x \quad$ **(Shown).**

(ii) Solving equations of the line and curve sumultaneously,

$$3 + 4\sin 2x = 5$$

$$\sin 2x = \frac{1}{2}$$

$$\Rightarrow \quad 2x = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{12}, \ \frac{5\pi}{12}$$

$\Rightarrow$ x-coordinate of A is $\dfrac{\pi}{12}$

and x-coordinate of C is $\dfrac{5\pi}{12}$ **(Ans).**

(iii) Equation of the curve: $y = 3 + 4\sin 2x$

Equation of the line: $y = 5$

Area of the shaded region

= area under the curve − area under the line

$$= \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left((3 + 4\sin 2x) - 5\right) \, dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (4\sin 2x - 2) \, dx$$

$$= \left[4\left(\frac{-\cos 2x}{2}\right) - 2x\right]_{\frac{\pi}{12}}^{\frac{5\pi}{12}}$$

$$= \left[-2\cos 2x - 2x\right]_{\frac{\pi}{12}}^{\frac{5\pi}{12}}$$

$$= \left(-2\cos 2\left(\frac{5\pi}{12}\right) - 2\left(\frac{5\pi}{12}\right)\right) - \left(-2\cos 2\left(\frac{\pi}{12}\right) - 2\left(\frac{\pi}{12}\right)\right)$$

$$= \left(-2\cos\left(\frac{5\pi}{6}\right) - \frac{5\pi}{6}\right) - \left(-2\cos\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right)$$

$$= -2\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}$$

$$= \sqrt{3} + \sqrt{3} - \frac{4\pi}{6}$$

$$= 2\sqrt{3} - \frac{2\pi}{3} = 1.369 \approx 1.37 \text{ unit}^2 \quad \textbf{(Ans).}$$

5 *(J10/P2/Q1)*

Find $\int \left(2 + 5x - \frac{1}{(x-2)^2}\right) dx$. [3]

Thinking Process

Integrate ✎ use $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

Solution

$$\int \left(2 + 5x - \frac{1}{(x-2)^2}\right) dx$$

$$= \int (2 + 5x - (x-2)^{-2}) \, dx$$

$$= 2x + 5\left(\frac{x^2}{2}\right) - \frac{(x-2)^{-1}}{-1} + C$$

$$= 2x + \frac{5}{2}x^2 + \frac{1}{x-2} + C \quad \textbf{(Ans).}$$

6 *(N10/P1/Q11)*

(i) Find $\int \frac{1}{\sqrt{1+x}} \, dx$. [2]

(ii) Given that $y = \frac{2x}{\sqrt{1+x}}$,

show that $\frac{dy}{dx} = \frac{A}{\sqrt{1+x}} + \frac{Bx}{\left(\sqrt{1+x}\right)^3}$, where A and B are to be found. [4]

(iii) Hence find $\int \frac{x}{\left(\sqrt{1+x}\right)^3} \, dx$

and evaluate $\int_0^3 \frac{x}{\left(\sqrt{1+x}\right)^3} \, dx$ [4]

Thinking Process

(i) Intergrate using $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$.

(ii) Differentiate ✎ use quotient rule, i.e.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - v'u}{v^2}.$$

(iii) Apply anti-differentiation:

$$\frac{d}{dx} f(x) = f'(x) \quad \Rightarrow \quad \int f'(x) dx = f(x)$$

Solution

(i) $\int \frac{1}{\sqrt{1+x}} \, dx = \int (1+x)^{-\frac{1}{2}} \, dx$

$$= \frac{(1+x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{(1+x)^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{1+x} + C \quad \textbf{Ans.}$$

(ii) $y = \frac{2x}{\sqrt{1+x}}$

$$\frac{dy}{dx} = \frac{(\sqrt{1+x})(2) - 2x\left(\frac{1}{2\sqrt{1+x}}\right)}{(\sqrt{1+x})^2}$$

$$= \left(2\sqrt{1+x} - \frac{x}{\sqrt{1+x}}\right) \div (\sqrt{1+x})^2$$

$$= \left(2\sqrt{1+x} - \frac{x}{\sqrt{1+x}}\right) \times \frac{1}{(\sqrt{1+x})^2}$$

$$= \frac{2\sqrt{1+x}}{(\sqrt{1+x})^2} - \frac{x}{(\sqrt{1+x})^3}$$

$$= \frac{2}{\sqrt{1+x}} - \frac{x}{(\sqrt{1+x})^3}$$

$$= \frac{2}{\sqrt{1+x}} + \frac{-x}{(\sqrt{1+x})^3}$$

$\Rightarrow \quad A = 2, \quad B = -1 \quad$ **Ans.**

(iii) Consider $\int \frac{x}{(\sqrt{1+x})^3}\,dx$

from part (ii), we have,

$$\frac{d}{dx}\left(\frac{2x}{\sqrt{1+x}}\right) = \frac{2}{\sqrt{1+x}} - \frac{x}{(\sqrt{1+x})^3}$$

$$\Rightarrow \quad \frac{x}{(\sqrt{1+x})^3} = \frac{2}{\sqrt{1+x}} - \frac{d}{dx}\left(\frac{2x}{\sqrt{1+x}}\right)$$

integrating both sides,

$$\int \frac{x}{(\sqrt{1+x})^3}\,dx = \int \frac{2}{\sqrt{1+x}}\,dx - \int \frac{d}{dx}\left(\frac{2x}{\sqrt{1+x}}\right)dx$$

$$= 2\int (1+x)^{-\frac{1}{2}}\,dx - \frac{2x}{\sqrt{1+x}}$$

$$= 2\left(\frac{(1+x)^{\frac{1}{2}}}{\frac{1}{2}}\right) - \frac{2x}{\sqrt{1+x}} + K$$

$$= 4\sqrt{1+x} - \frac{2x}{\sqrt{1+x}} + K$$

now, $\int_0^3 \frac{x}{\left(\sqrt{x+1}\right)^3}\,dx$

$$= \left[4\sqrt{1+x} - \frac{2x}{\sqrt{1+x}}\right]_0^3$$

$$= \left(4\sqrt{4} - \frac{6}{\sqrt{4}}\right) - (4\sqrt{1} - 0)$$

$$= 8 - 3 - 4 = 1 \quad \textbf{Ans.}$$

7 *(N10/P1/Q12 Either)*

A curve is such that $\frac{dy}{dx} = 4x^2 - 9$. The curve passes through the point (3, 1).

(i) Find the equation of the curve. [4]

The curve has stationary points at A and B.

(ii) Find the coordinates of A and of B. [3]

(iii) Find the equation of the perpendicular bisector of the line AB. [4]

Thinking Process

(i) Integrate $\frac{dy}{dx}$ function to obtain equation of the curve.

(ii) To find stationary points, equate $\frac{dy}{dx}$ to 0.

(iii) To find equation of ⊥ bisector of *AB* ✎ find the gradient of ⊥ bisector of *AB*. ✎ Find the mid-point of *AB*.

Solution

(i) $\frac{dy}{dx} = 4x^2 - 9 \quad \Rightarrow \quad dy = (4x^2 - 9)\,dx$

integrate both sides

$$\int dy = \int (4x^2 - 9)\,dx$$

$$\Rightarrow \quad y = 4\left(\frac{x^3}{3}\right) - 9x + C$$

the curve passes througn (3, 1)

$$\Rightarrow \quad 1 = 4\left(\frac{3^3}{3}\right) - 9(3) + C$$

$$1 = 36 - 27 + C$$

$$\Rightarrow \quad 1 = 9 + C \quad \Rightarrow \quad C = -8$$

∴ equation of curve is: $y = \frac{4}{3}x^3 - 9x - 8$ **Ans.**

(ii) For stationary points, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 4x^2 - 9 = 0 \quad \Rightarrow \quad x^2 = \frac{9}{4} \quad \Rightarrow \quad x = \pm\frac{3}{2}$$

From part (i), equation of curve is:

$$y = \frac{4}{3}x^3 - 9x - 8$$

when $x = \frac{3}{2}$, $\quad y = \frac{4}{3}\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right) - 8$

$$= \frac{4}{3}\left(\frac{27}{8}\right) - \frac{27}{2} - 8$$

$$= \frac{9}{2} - \frac{27}{2} - 8 = -17$$

∴ point $A\left(\frac{3}{2}, -17\right)$ **Ans.**

When $x = -\frac{3}{2}$, $\quad y = \frac{4}{3}\left(-\frac{3}{2}\right)^3 - 9\left(-\frac{3}{2}\right) - 8$

$$= \frac{4}{3}\left(\frac{-27}{8}\right) + \frac{27}{2} - 8$$

$$= -\frac{9}{2} + \frac{27}{2} - 8 = 1$$

∴ point $B\left(-\frac{3}{2}, 1\right)$ **Ans.**